Student Name: SOLUTIONS

MATHEMATICS: SPECIALIST

Question/Answer Booklet – Section 1 – Calculators *NOT* allowed – Notes sheets *NOT* allowed

Time allowed for this paper

Section	Reading	Working
Calculator-free	5 minutes	50 minutes
Calculator-assumed	10 minutes	100 minutes

Materials required/recommended for this paper

Section One (Calculator-free): 52 marks

To be provided by the supervisor
Section One Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, highlighter, eraser, ruler, correction fluid/tape

Special items: nil

Section Two (Calculator-assumed): 98 marks

To be provided by the supervisor
Section Two Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the School Curriculum and Standards Authority for this course.

Important Note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

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Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare answer pages may be found at the end of this booklet. If you need to use them, indicate in the original answer space where the answer is continued (i.e. give the page number).

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Structure of this paper

Questions	Marks available	Your score
1	4	
2	7	
3	8	
4	10	
5	12	
6	7	
7	4	
Total for Section 1	**52**	
Total for Section 2	**98**	
Total marks = 150		
		%

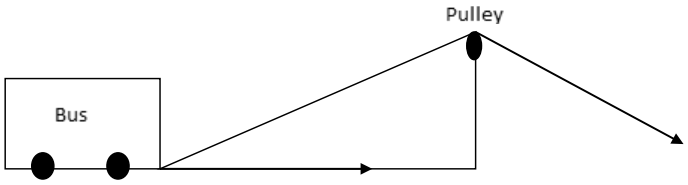
Section One: Calculator-free **(50 Marks)**

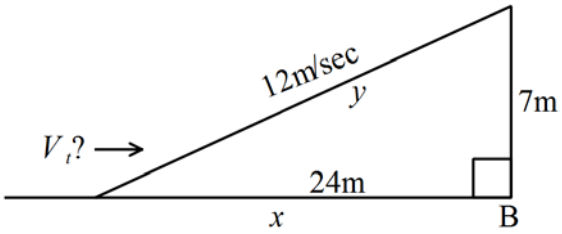
This section has **nine (9)** questions. Answer **all** questions. Write your answers in the space provided.

Suggested working time for this section is 50 minutes.

Question 1 **(4 marks)**

A bus is drawn along a horizontal road by a rope which passes through a pulley 7m vertically above a point B on the road. If the rope is wound in at 12 m/sec, find the velocity of the bus at the instant when the bus is 24m from B.





i.e. find $\frac{dy}{dx}$ when $x=24$

$y^2-x^2=7^2$

✓ establishes *x*-*y* relationship

$\frac{d}{dt}\left(y^2-x^2\right)=\frac{d}{dt}(49)$

$2y\frac{dy}{dt}-2x\frac{dx}{dt}=0$

$x\frac{dx}{dt}=y\frac{dy}{dt}$

✓ determines differential *x*-*y* relationship

from $y^2-x^2=49$, when $x=24$

$y=\pm 25$ (reject the -ve)

✓ calculates *y* value for *x*=24

$\therefore 24\frac{dx}{dt}=25(12)$

$\frac{dx}{dt}=\frac{25\times 12}{24}=12\cdot 5$ m/sec

✓ calculates required rate

OR

$\frac{dy}{dt}=12$

$y=\sqrt{x^2+7^2}$

✓ establishes *x*-*y* relationship

$y=\sqrt{x^2+49}$

$y=\left(x^2+49\right)^{1/2}$

$\frac{dy}{dx}=\frac{1}{2}\left(x^2+49\right)^{-1/2}\cdot 2x$

✓ determines differential *x*-*y* relationship

$\frac{dx}{dt}=\frac{dx}{dy}\cdot\frac{dy}{dt}$

$=\frac{\sqrt{x^2+49}}{x}\cdot 12$ at $x=24$

✓ calculates *y* value for *x*=24

$=\frac{\sqrt{24^2+49}}{24}\cdot 12$

$=12\cdot 5$ m/sec

✓ calculates required rate

Question 2 **(7 marks)**

Consider the curve represented by $x^2 - xy + \frac{3}{2}y^2 = 9$.

(a) Find the gradient of the curve at any point (*x*, *y*). (2 marks)

$2x - y - x\frac{dy}{dx} + 3y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(3y - x) = y - 2x$ ✓

$\frac{dy}{dx} = \frac{y - 2x}{3y - x}$

✓

(b) Find the equation of the tangent to the curve at the point (3,0) **and** find the equation of the tangent to the curve at the point $(0, \sqrt{6})$. Write each equation in the form $y = ax + b$. (3 marks)

at (3,0) $\frac{dy}{dx} = \frac{y - 2x}{3y - x} = \frac{0 - 6}{0 - 3} = 2$

$y = 2x + c$ ✓

$(3,0) \Rightarrow 0 = 6 + c \Rightarrow c = -6$

$y = 2x - 6$

✓

at $(0,\sqrt{6})$ $\frac{dy}{dx} = \frac{\sqrt{6} - 0}{3\sqrt{6} - 0} = \frac{1}{3}$

$y = \frac{1}{3}x + \sqrt{6}$

✓

(c) Find the acute angle between the tangent to the curve at the point (3,0) and the tangent to the curve at the point $(0, \sqrt{6})$. Give your answer in the form $k\pi$ where k is a real constant. (2 marks)

Let the angle formed between first line and horizontal line be α and the angle formed between second line and horizontal line be β. The angle we want then is $\alpha - \beta$. $\text{Tan}\,\alpha = 2$ and $\text{Tan}\,\beta = \frac{1}{3}$ ✓

$\text{Tan}(\alpha - \beta) = \frac{\text{Tan}\,\alpha - \text{Tan}\,\beta}{1 + \text{Tan}\,\alpha\,\text{Tan}\,\beta}$ $\quad\quad$ $\text{Tan}(\alpha - \beta) = 1 \Rightarrow (\alpha - \beta) = \frac{\pi}{4}$

$\text{Tan}(\alpha - \beta) = \frac{2 - \frac{1}{3}}{1 + 2\left(\frac{1}{3}\right)} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$

✓

Question 3 **(8 marks)**

(a) If $z = 3 - 4i$, determine the reciprocal, $\frac{1}{z}$. (2 marks)

$\frac{1}{3-4i} \times \frac{3+4i}{3+4i}$

$\frac{3+4i}{9+16}$ ✓

$\frac{3+4i}{25}$

✓

(b) Let the non-zero complex number $z = a + bi$. Show that $\frac{1}{a+bi} = \frac{\overline{z}}{|z|^2}$. (3 marks)

$LHS: \frac{1}{a+bi} \times \frac{a-bi}{a-bi}$

$\frac{a-bi}{a^2+b^2}$ ✓✓

$\frac{\overline{z}}{|z|^2}$ as required

✓

(c) A complex number z satisfies the equation $z - 3|z|^2 = i - 3$, find the complex number.
Hint: Let $z = a + bi$ (3 marks)

$a + bi - 3(a^2 + b^2) = i - 3$

$(a - 3a^2 - 3b^2) + bi = -3 + i$

$\text{Im}: b = 1$

$\text{Re}: a - 3a^2 - 3(1)^2 = -3$ ✓

$a - 3a^2 = 0$

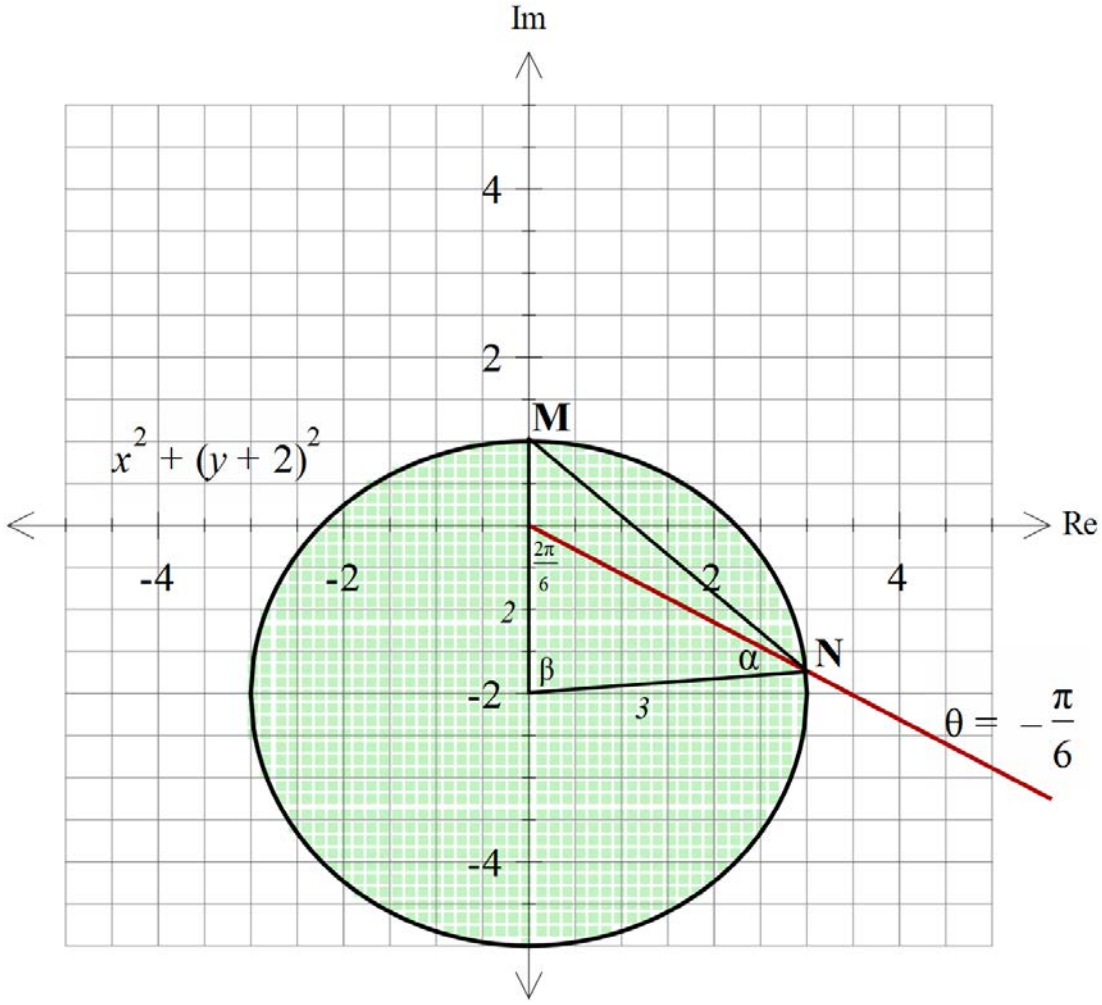
$a(1 - 3a) = 0$

$a = 0 \text{ or } a = \frac{1}{3}$

✓✓

Question 4 **(10 marks)**

(a) On the Argand diagram below sketch the locus of points defined by: $|z+2i| \leq 3$ [2]



✓ draws a circle with a solid line and ✓ shades inside

(b) Describe the locus drawn above; [2]

All points located within and on the circle, centre (0, –2) with radius 3.

✓✓ states is a circle with center (0,– 2) and radius 3

(c) M is the point with the maximum Im(z) in the locus.
Clearly label the point M on your locus above.

[1]

M: $(0, i)$ ✓ correctly labels the point M

(d) Label the point of intersection between $|z+2i| = 3$ and the locus of $\theta = -\frac{\pi}{6}$ as the point N on your locus above. [1]

✓ correctly labels the point N

continued overleaf / …

Question 4 (*continued*) **(10 marks)**

(e) Determine a value for the distance between the points M and N to the nearest unit. [4]

HINTS: $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx \frac{\pi}{5}$ $\cos\left(\frac{7\pi}{15}\right) \approx 0\cdot1$

from diagram

$$\frac{\sin\alpha}{2} = \frac{\sin\frac{2\pi}{6}}{3}$$

$$\alpha = \sin^{-1}\left(\frac{2}{3}\times\frac{\sqrt{3}}{2}\right) = \sin^{-1}\frac{1}{\sqrt{3}} \approx \frac{\pi}{5} \quad \text{(given)}$$

✓ uses Sine rule to find α value

$$\beta = \pi - \frac{2\pi}{6} - \frac{\pi}{5} = \frac{7\pi}{15}$$

✓ calculates β value

Thus $MN^2 = 3^2 + 3^2 - 2\times3\times3\times\cos\left(\frac{7\pi}{15}\right)$

✓ uses cosine rule to find MN

i.e. $MN^2 \approx 18 - 18\times0\cdot1 = 16\cdot2$

$\therefore\ MN \approx \sqrt{16\cdot2} \approx 4$

✓ calculates MN to the nearest unit

from diagram

$$3^2 = 2^2 + x^2 - 2(2)(x)\cos\frac{\pi}{3}$$

$$9 = 4 + x^2 - 4x\bullet\frac{1}{2}$$

$$0 = x^2 - 2x - 5$$

$$0 = (x-1)^2 - 6$$

$\therefore\ x = \pm\sqrt{6} + 1$ reject – ves - distance positive

✓ uses Cosine rule to find x values

$$y^2 = 1^2 + \left(\pm\sqrt{6}+1\right)^2 - 2\left(\pm\sqrt{6}+1\right)\cos\frac{2\pi}{3}$$

$$y^2 = 1^2 + 6 \pm 2\sqrt{6} + 1 \pm \sqrt{6} + 1$$

$$y^2 = 9 \pm 3\sqrt{6}$$

$y = \pm\sqrt{9 \pm 3\sqrt{6}}$ reject – ves - distance positive

✓ uses Cosine rule to find y values

i.e. $y \approx 4$ or $1\cdot28$ reject $1\cdot28$ – wrong direction

✓ selects appropriate y value

✓ answers to the nearest unit

Question 5 **(12 marks)**

(a) If $h(x) = x^4 + ax^3 + 3x^2 + bx - 10$ has a factor (x-1) and a remainder of 36 when divided by (x-2).

(i) Determine the values of a and b (3 marks)

$h(1) = 0 \Rightarrow a + b = 6 \Rightarrow b = 6 - a$

$h(2) = 36 \Rightarrow 4a + b = 9$

$4a + 6 - a = 9$

$a = 1 \quad b = 5$

✓✓✓

(ii) Fully factorise $h(x)$ (3 marks)

$$(x-1)\overline{)x^4 + x^3 + 3x^2 + 5x - 10} \quad x^3 + 2x^2 + 5x + 10$$

let $g(x) = (x^3 + 2x^2 + 5x + 10)$

$g(-2) = 0 \Rightarrow (x+2)$ is a factor

✓✓✓

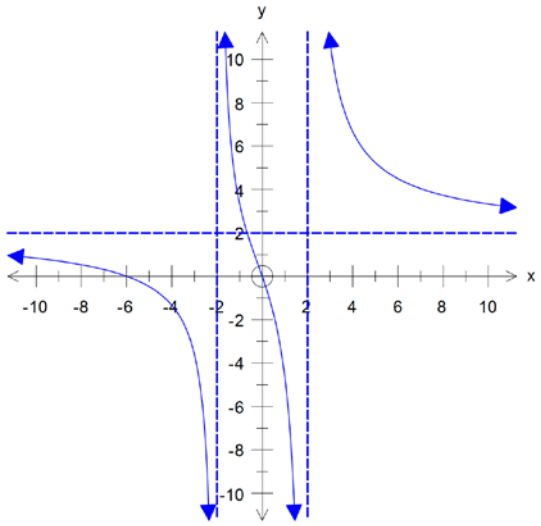
$$(x+2)\overline{)(x^3 + 2x^2 + 5x + 10)} \quad x^2 + 5$$

$x^4 + x^3 + 3x^2 + 5x - 10 = (x-1)(x+2)(x^2+5)$

(b) The graph of $y = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$ is shown on the axes.

Determine the value of the constants a, b, c, d, e and f. (6 marks)

By inspection of the horizontal asymptote $y = 2$ we know $a = 2$ and $d = 1$ ✓✓



By inspection of the horizontal asymptote at $y = 2$ we know $a = 2$ and $d = 1$ ✓✓

By inspection of the vertical asymptotes at $x = \pm 2$ we know the denominator would be

$(x-2)(x+2) = x^2 - 4 \Rightarrow e = 0$ and $f = -4$ ✓✓

Using the point (0,0) we get

$$0 = \frac{0+0+c}{0-4}$$ ✓

$$0 = c$$

Using another point e.g. (-6,0) we get

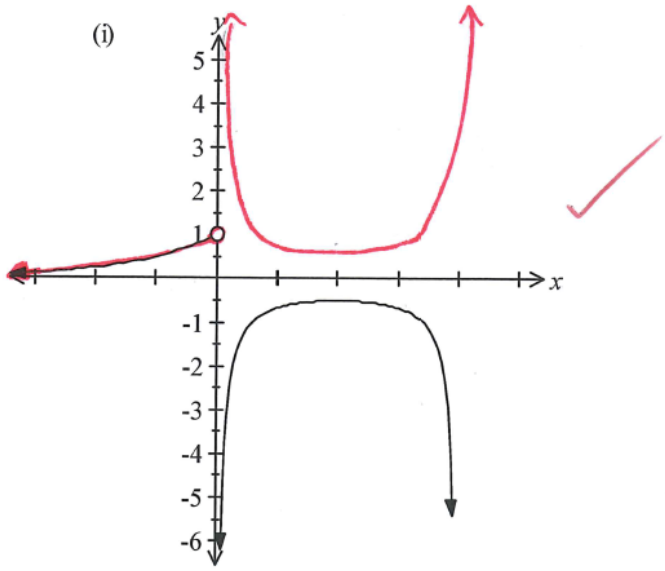
$$0 = \frac{2(-6)^2 + b(-6)}{(-6)^2 - 4}$$

$0 = 72 - 6b \Rightarrow b = 12$ ✓

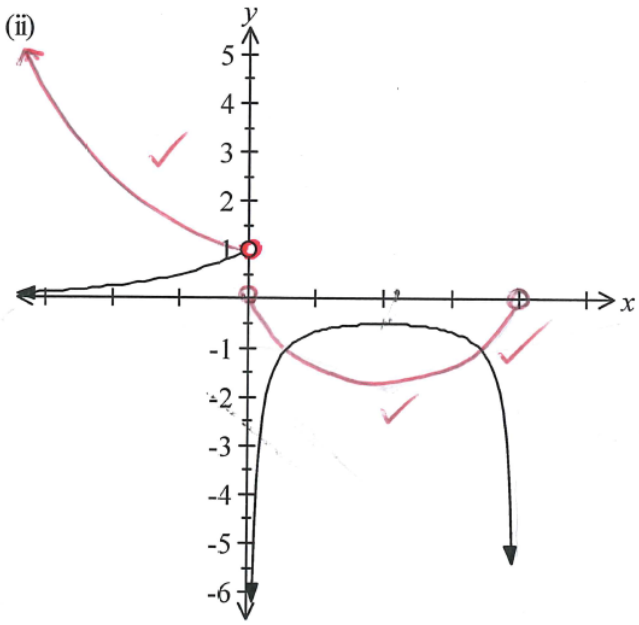
Question 6 **(7 marks)**

(a) The graph of $y = f(x)$ is drawn on the 3 sets of axes drawn below. Neatly sketch the graph of the following

(i) $y = |f(x)|$ on the first set of axes (1 mark)



(ii) $y = \dfrac{1}{f(x)}$ on the second set of axes. (3 marks)



(b) Given $\mathbf{g(x)} = \sqrt{1-x^2}$ determine $\mathbf{g^{-1}(x)}$ and state the domain and range of $\mathbf{g^{-1}(x)}$ given that $\mathbf{g^{-1}(x)}$ is a function. (3 marks)

$y = \sqrt{1-x^2}$

$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$y^2 = 1-x^2$

$y = \pm\sqrt{1-x^2}$

$g^{-1}(x) = \sqrt{1-x^2}$ ✓

As $g^{-1}(x)$ is a function the $\pm$ must go.

$g(x) = \sqrt{1-x^2}$ $D: -1 \le x \le 1$ $R: 0 \le y \le 1$ ✓

$g^{-1}(x) = \sqrt{1-x^2}$ switch the Domain and Range $D: 0 \le x \le 1$ $R: -1 \le y \le 1$ however the range must be restricted to be $R: 0 \le y \le 1$ to allow it to be a function ✓

Question 7 **(4 marks)**

Find the volume generated when the region bounded by the of $y = 2x^2 - 3$, the line $y = 5$ and the y-axis is rotated about the y-axis.

$y = 2x^2 - 3$

$\frac{y+3}{2} = x^2$

$\int_{-3}^{5} \pi x^2 dy$

$\pi \int_{-3}^{5} \frac{y+3}{2} dy$

$\pi \left[\frac{y^2}{4} + \frac{3y}{2} \right]_{-3}^{5}$ ✓✓

$\pi \left[\left(\frac{5^2}{4} + \frac{3(5)}{2} \right) - \left(\frac{(-3)^2}{4} + \frac{3(-3)}{2} \right) \right]$

$\pi \left(\frac{55}{4} + \frac{9}{4} \right)$

$16\pi \text{ units}^3$

✓✓

END OF QUESTIONS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Additional working space

Question number(s): _____

Additional working space

Question number(s): _____

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Additional working space

Question number(s): _____